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Quasi-periodic continued fractions

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Abstract

Using recent work of Adamczewski and Bugeaud, we are able to relax the conditions given by Baker to establish transcendence in the class of quasi-periodic continued fractions.

Keywords: Continued fraction; Quasi-periodic; Transcendence; Subword complexity

1. Introduction

In 1962 A. Baker [5] proved a couple of theorems concerning the transcendence of quasi-periodic continued fractions. In order to state what he proved we will use the notation now common in Formal Language theory.

Let Σ denote a finite set of positive integers and Σ^* represent the set of finite words over Σ. Σ^* is a monoid under concatenation with the identity being the empty word ϵ. If $\mathbf{u} = (u_n)_{n\geqslant 1}$ is an infinite sequence with $u_n \in \Sigma$ for $n \geqslant 1$ then $\mathbf{u}$ can also be considered as an infinite word $\mathbf{u} = u_1u_2u_3\dots u_n\dots$ over Σ. Σ^ω denotes the set of infinite words over Σ. $\alpha = [0, \mathbf{u}]$ is the associated continued fraction to $\mathbf{u}$.

We can now state the version of Theorem 3 in [5] which we will improve upon in this paper.

Theorem 1. *(See [5].) Suppose $(\lambda_n)_{n\geqslant 1}$ is a sequence of positive integers, $\delta > 0$, and $\lambda_n > (2+\delta)\lambda_{n-1}$ for all $n \geqslant 2$. If $\{w_1, w_2, \dots, w_k\}$ is a set of non-empty words over Σ and $\mathbf{u} = w_1^{\lambda_1} w_2^{\lambda_2} \dots w_k^{\lambda_k} w_1^{\lambda_{k+1}} w_2^{\lambda_{k+2}} \dots$, then the continued fraction $\alpha = [0, \mathbf{u}]$ is either a quadratic irrational or transcendental.*

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The proofs of his results involved applying the extension of Roth's theorem [11] which limits the accuracy of the approximation to an algebraic number by elements of a particular quadratic field, and seemed therefore to be the best one could do in this situation.

Recently Adamczewski and Bugeaud [1] have developed a new technique, based on the use of the Schmidt Subspace Theorem, which will be used here to give a substantial improvement to Baker's result. Specifically we can relax the given condition to $\lambda_n > (\tau + \delta)\lambda_{n-1}$ for $n \geqslant 2$ where $\tau = \frac{1}{2}(1 + \sqrt{5})$. Also in the special case of $k = 2$ we can further relax the condition to $\lambda_n > (\sqrt{2} + \delta)\lambda_{n-1}$ for $n \geqslant 2$. It should be noted that Baker [5] did obtain $\sqrt{2} + \delta$ for $k = 2$ in the special case that $a_1 \gg_\delta a_2$. (a_1 and a_2 are the two distinct partial quotients.)

Our proof will depend on changing (slightly) the condition $(**)_{w,w'}$ of [1] and deriving good estimates for the size of denominators of certain convergents to the given continued fraction. This will involve working with continuants and using matrix theory associated with continued fractions. Such techniques have occurred sporadically in recent years with different notation and emphasis. The reader can consult [6–8,10] for some related work.

In Section 2 we discuss the class of quasi-periodic infinite words. In Section 3 we will consider continuants and our transcendence criterion derived from [1]. In Section 4 we prove our main Theorem and give examples of infinite words with sub-affine complexity for which the current technique is not effective.

Since this paper was written, a recent improvement was obtained by Adamczewski and Bugeaud [2] thanks to a different use of the Schmidt Subspace Theorem. In particular, it allows us to replace the value $(1 + \sqrt{5})/2$ in Theorem 13 by 1.

2. Quasi-periodic words

As before, Σ will denote a fixed set of positive integers. If $x \in \Sigma^*$, then $|x|$ denotes the length of the word x. Also x^* denotes the set $\{\epsilon, x, x^2, x^3, \ldots\}$. Our formal definition of what is meant by quasi-periodic is contained in the following.

Definition 2. Suppose $\{w_1, w_2, \ldots, w_k\}$ is a finite set of non-empty words over Σ and suppose $(\lambda_n)_{n\geqslant 1}$ is a sequence of positive integers with $\lambda_n \to \infty$. Let $\mathbf{u}_\lambda\ (w_1, w_2, \ldots, w_k) = w_1^{\lambda_1} w_2^{\lambda_2} \ldots w_k^{\lambda_k} w_1^{\lambda_{k+1}} \ldots$. Then $\mathbf{u}_\lambda \in \Sigma^w$ is said to be a quasi-periodic infinite word.

The ultimately periodic infinite words play a special role in this area, as their associated continued fractions correspond exactly to quadratic irrationals. It seems useful to characterize those quasi-periodic words which are ultimately periodic. The result is not surprising, but appears to take a bit more effort than expected.

We will use some of the standard definitions and results from [12]. In particular, $y \in \Sigma^* \setminus \{\epsilon\}$ is said to be primitive if $y = z^n$ for some $n \geqslant 1$ implies that $n = 1$. Also, two words x and y are conjugate if there is a word z such that $xz = zy$.

Proposition 3. *If* $\mathbf{u} = \mathbf{u}_\lambda\ (w_1, w_2, \ldots, w_k)$ *is ultimately periodic, then there exists a non-empty word* $z \in \Sigma^*$ *such that* $w_i \in z^*$ *for* $1 \leqslant i \leqslant k$.

Proof. We can write $\mathbf{u} = xy^\infty = xyyy\ldots$ where $x \in \Sigma^*$ and y is primitive. Since $\lambda_n \to \infty$ we can find an integer N satisfying both of the following conditions.

(a) $$\sum_{n=0}^{N-1}\sum_{j=1}^{k}\lambda_{nk+j}|w_j| > |x|;$$

(b) $$\lambda_{nk+i}|w_i| \geqslant 3|y| \quad \text{for } 1 \leqslant i \leqslant k \text{ and } n \geqslant N.$$

Thus for $n \geqslant N$ and for $1 \leqslant i \leqslant k$ there are positive integers $h_{i,n}$ and words $r_{i,n}$, $s_{i,n}$ with $0 \leqslant |r_{i,n}|, |s_{i,n}| < |y|$, such that

$$w_i^{\lambda_{kn+i}} = r_{i,n} y^{h_{i,n}} s_{i,n}.$$

Since for each i, $1 \leqslant i \leqslant k$ there are only a finite number of possibilities for the pairs $(r_{i,n}, s_{i,n})_{n \geqslant N}$, we can find words r_i, s_i, positive integers d_i, e_i f_i, g_i with $f_i > d_i$, $g_i > e_i$ such that $w_i^{d_i} = r_i y^{e_i} s_i$ and $w_i^{f_i} = r_i y^{g_i} s_i$.

Hence $r_i y^{g_i} s_i = w_i^{d_i} w_i^{f_i - d_i} = r_i y^{e_i} s_i w_i^{f_i - d_i}$.

Simplifying and letting $b_i = g_i - e_i$, $c_i = f_i - d_i$, we get

$$y^{b_i} s_i = s_i w_i^{c_i}.$$

Also $|s_i| < |y|$, so s_i is a prefix of y: $y = s_i u_i$. Replace y in the previous equation then gives

$$(u_i s_i)^{b_i} = w_i^{c_i}.$$

By Proposition 1.3.1 of [12], there exists a primitive word z_i with $w_i \in z_i^*$ and $u_i s_i \in z_i^*$. But $u_i s_i$ is conjugate to $s_i u_i = y$, which is primitive and so $u_i s_i$ is also primitive which implies that $z_i = u_i s_i$.

Finally we note that $w_1^{\lambda_{nk+1}} w_2^{\lambda_{nk+2}} \dots w_k^{\lambda_{nk+k}}$ is a subword of y^∞, so that for $1 \leqslant i \leqslant k-1$ we obtain $s_i u_{i+1} = y = s_i u_i$. Thus $u_1 = u_2 = \cdots = u_k$ and $z_i = z$ (say) for $1 \leqslant i \leqslant k$. □

It is known that two words w_1, w_2 commute if and only if they are powers of the same word [12]. Thus, an equivalent condition on $\{w_1, w_2, \dots, w_k\}$ to produce an ultimately periodic word is that they form a commuting set of words. We will sometimes refer to the ultimately periodic class as the trivial class of quasi-periodic infinite words.

For reasons that will become clear we have adopted a slightly different version of what Adamczewski and Bugeaud refer to as condition $(**)_{w,w'}$.

Definition 4. Suppose $w', w'' > 0$. We say $\mathbf{u} \in \Sigma^\omega$ satisfies condition (w', w'') if all the following hold:

(a) $\mathbf{u}$ is not ultimately periodic;
(b) there exist three sequences of finite words $(U_n)_{n \geqslant 1}$, $(V_n)_{n \geqslant 1}$, $(W_n)_{n \geqslant 1}$, such that for any $n \geqslant 1$:
  (i) $U_n V_n W_n$ is a prefix of $\mathbf{u}$;
  (ii) $|U_n| \leqslant w'|V_n|$;
  (iii) W_n is a prefix of V_n^s for some positive integer s;
  (iv) $|W_n| \geqslant w''|V_n|$;
  (v) $|U_n| \to \infty$ as $n \to \infty$.

We will put $r_n = |U_n|$, $s_n = |V_n|$, $t_n = |W_n|$. Note that the given conditions (ii), (iv) and (v) imply that $s_n \to \infty$ and $t_n \to \infty$.

Remark 5. The class of infinite words which satisfies the set of conditions modified by putting $w' = 0$, $U_n = \epsilon$ for all n, and (v) removed has a much more complete and satisfactory theory as proved in [1].

However none of the non-trivial quasi-periodic infinite words belong to this more restrictive class.

Lemma 6. *Suppose $1 < \beta < \gamma < \infty$ and let $(\lambda_n)_{n\geqslant 1}$ be a sequence of positive integers satisfying $\beta\lambda_{n-1} < \lambda_n < \gamma\lambda_{n-1}$ for $n \geqslant 2$. If $k \geqslant 2$ and $\{w_1, w_2, \dots, w_k\}$ is not a commuting set of non-empty words over Σ, put*

$$w' = \left(\beta^k - 1\right)^{-1} \quad \textit{and} \quad w'' = \left(\sum_{i=0}^{k-1} \gamma^i \cdot \max_{1\leqslant i\leqslant k} |w_i|\right)^{-1}.$$

Then $\mathbf{u} = \mathbf{u}_{\boldsymbol{\lambda}}\ (w_1, w_2, \dots, w_k)$ satisfies condition (w', w'').

Proof. By Proposition 3 and the comments in the paragraph following we obtain the fact that $\mathbf{u}$ is not ultimately periodic.

Set

$$\begin{aligned} U_n &= w_1^{\lambda_1} w_2^{\lambda_2} \dots w_{k-2}^{\lambda_{nk-2}}; \\ V_n &= w_{k-1}^{\lambda_{nk-1}} w_k^{\lambda_{nk}} \dots w_{k-2}^{\lambda_{nk+k-2}} w_{k-1}^{\lambda_{nk+k-1}-\lambda_{nk-1}}; \\ W_n &= w_{k-1}^{\lambda_{nk-1}} w_k^{\lambda_{nk}}. \end{aligned}$$

(If $k = 2$, then w_0 is assumed to be w_2.)

We will show that $(U_n, V_n, W_n)_{n\geqslant 1}$ satisfies the five conditions under part (b) of Definition 4.

(i) $U_n V_n W_n$ is clearly a prefix of $\mathbf{u}$.

(ii) $r_n = |U_n| = \sum_{i=1}^{k} X_i |w_i|$, where

$$X_i = \begin{cases} \sum_{j=0}^{n-1} \lambda_{jk+i} & \text{if } 1 \leqslant i \leqslant k-2, \\ \sum_{j=0}^{n-2} \lambda_{jk+i} & \text{if } k-1 \leqslant i \leqslant k. \end{cases}$$

Also $s_n = |V_n| = \lambda_{nk}|w_k| + \sum_{i=1}^{k-1} \lambda_{nk+i}|w_i|$.

Now, by assumption, $\lambda_{i-k} < \frac{1}{\beta^k}\lambda_i$ for $i > k$. Thus, for $1 \leqslant i \leqslant k-2$, we obtain

$$X_i < \left(\frac{1}{\beta^k} + \frac{1}{\beta^{2k}} + \dots + \frac{1}{\beta^{nk}}\right)\lambda_{nk+i} < \frac{1}{\beta^k - 1}\lambda_{nk+i}, \quad \text{and}$$

$$X_{k-1} < \frac{1}{\beta^k - 1}\lambda_{nk-1}; \qquad X_k < \frac{1}{\beta^k - 1}\lambda_{nk}.$$

Hence $r_n < \frac{1}{\beta^k - 1} s_n$, as required.

(iii) W_n is a prefix of V_n.
(iv) $t_n = |W_n| > \lambda_{nk}|w_k|$, so

$$\frac{t_n}{s_n} > \frac{\lambda_{nk}|w_k|}{\lambda_{nk}|w_k| + \sum_{i=1}^{k-1}\lambda_{nk+i}|w_i|}.$$

But $\lambda_{nk+i} < \gamma^i\lambda_{n-k}$ for $i \geqslant 1$, and thus

$$\frac{t_n}{s_n} > \frac{|w_k|}{|w_k| + \sum_{i=1}^{k-1}\gamma^i|w_i|} > w''.$$

(v) $r_n \to \infty$ since $\lambda_n \to \infty$. □

Quasi-periodic infinite words also have the interesting property that many of them have very modest subword complexity. Recall that the complexity $p(n)$ of an infinite word $\mathbf{u}$ is the number of distinct blocks of length n occurring in $\mathbf{u}$. For a good reference consult [3]. Purely periodic infinite words have a bounded complexity function. The following shows examples of sub-affine complexity within the class of quasi-periodic words. Our proof goes along the same lines as [9]. We say a word w is a right special factor of $\mathbf{u}$ if both wa and wb are subwords of $\mathbf{u}$.

Proposition 7. *Suppose $(\lambda_m)_{m\geqslant 1}$ is a sequence of positive integers satisfying $\lambda_m > \beta\lambda_{m-1}$ for all $m \geqslant 2$ and a fixed $\beta > 1$. Suppose also a, b are distinct integers from Σ. If $\mathbf{u} = \mathbf{u}_\lambda\ (a, b)$ then $p(n) \leqslant (3 + \lfloor\frac{\log 2}{\log\beta}\rfloor)n$ for $n \geqslant 1$.*

Proof. Let $n \geqslant 1$. Then $p(n+1) - p(n)$ is the number of words w in $\mathbf{u}$, of length n, which are right special factors. Clearly a^n and b^n are right special factors. Also if $\lambda_m < n < \lambda_m + \lambda_{m-1}$ then the word $a^{n-\lambda_m}b^{\lambda_m}$, if m is even, or the word $b^{n-\lambda_m}a^{\lambda_m}$, if m is odd, are right special factors and these are the only possibilities. Let $m_0 = \min\{m\colon n \leqslant \lambda_m + \lambda_{m-1}\}$ and $t_0 = \max\{t\colon \lambda_{m_0+t} < n\}$. Thus $p(n+1) = p(n) + 3 + t_0$. Note that $t_0 \geqslant -1$. Furthermore, $t > \frac{\log 2}{\log\beta}$ implies that $\beta^t > 2$ and so

$$\lambda_{m_0+t} > \beta^t\lambda_{m_0} > 2\lambda_{m_0} > \lambda_{m_0} + \lambda_{m_0-1}.$$

Hence $t_0 \leqslant \lfloor\frac{\log 2}{\log\beta}\rfloor$, and the result is established. □

3. Continuants and the transcendence criterion

If $\mathbf{u}$ is an infinite word over Σ then we can form the associated continued fraction $\alpha = [0, \mathbf{u}]$. If α has convergents $(\frac{p_n}{q_n})_{n\geqslant 0}$ and $\mathbf{u}$ satisfies a condition (w', w'') then the transcendence criterion of [1] is the requirement that there exists an $\eta > 0$ such that for all sufficiently large n we have

$$(q_{r_n}q_{r_n+s_n})^{1+\eta} < q_{r_n+s_n+t_n}.$$

In order for us to prove this condition under certain hypothesis we have to consider the growth of certain continuants. Some aspects of this work have appeared in [6,10].

Definition 8. Mat: $\Sigma^* \to \mathbb{Z}^+$ is defined by setting $\text{Mat}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{Mat}(a) = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$ for all $a \in \Sigma$, and recursively putting $\text{Mat}(aw) = \text{Mat}(a)\,\text{Mat}(w)$ for $a \in \Sigma$, $w \in \Sigma^+$. □

If $w = a_1 a_2 \ldots a_n$ and we put $\frac{p_r}{q_r} = [0, a_1, a_2, \ldots, a_r]$ for $1 \leqslant r \leqslant n$ then $\text{Mat}(w) = \begin{bmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{bmatrix}$. Since $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ it is clear that if $\mathcal{W} = \text{Mat}(w)$ then the spectral radius, $\rho(\mathcal{W})$, equals

$$\frac{1}{2}\big(q_n + p_{n-1} + \sqrt{(q_n + p_{n-1})^2 + 4(-1)^{n-1}}\big).$$

If $w = a_1 a_2 \ldots a_n$ then the reverse word is the word $w^R = a_n a_{n-1} \ldots a_1$. Then we have $\text{Mat}(w^R) = (\text{Mat}(w))^T$, where $(\cdot)^T$ denotes the transposed matrix.

Definition 9. The continuant $K : \Sigma^* \to \mathbb{Z}^+$ is defined by setting $K(w) = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{Mat}(w) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We can then prove directly from this definition the following properties of the continuant [6,13].

(a) $K(\lambda) = 1$.
(b) $K(a) = a$ for all $a \in \Sigma$.
(c) $K(abw) = aK(bw) + K(w)$ for $a, b \in \Sigma$, $w \in \Sigma^*$.
(d) $K(w^R) = K(w)$.
(e) $K(w_1 a b w_2) = K(w_1 a)K(b w_2) + K(w_1)K(w_2)$.
(f) $K(a_1 a_2 \ldots a_n) = q_n$ where $\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n]$.
(g) If v is a proper prefix of w, then $K(v) < K(w)$.
(h) $K(w_1)K(w_2) \leqslant K(w_1 w_2) < 2K(w_1)K(w_2)$ for $w_1, w_2 \in \Sigma^*$.

We also need the following lemma. Baxa [6] proved a special case of (b).

Lemma 10. *If $w \in \Sigma^+$ and $\mathcal{W}$ =Mat(w), then*

(a) $K(w) < \rho(\mathcal{W}) < 2K(w)$.
(b) $m \log \rho(\mathcal{W}) - \log 2 < \log K(w^m) < m \log \rho(\mathcal{W})$ *for* $m \geqslant 1$.
(c) *If $|w| \geqslant 2$ there exist constants C, D, depending only on Σ such that $C|w| \leqslant \log K(w) < D|w|$.*

Proof. (a) If $w = a_1 a_2 \ldots a_n$ then continuant property (f) shows that $K(w) = q_n$. Also $\rho(\mathcal{W}) = \frac{1}{2}(q_n + p_{n-1} + \sqrt{(q_n + p_{n-1})^2 + 4(-1)^{n-1}})$. If $n = 1$ the result is clear. If $n \geqslant 2$ then we use the fact that $q_n \geqslant p_{n-1} + 1$.

(b) From part (a): $K(w^m) < \rho(\mathcal{W}^m) < 2K(w^m)$. But $p(\mathcal{W}^m) = (p(\mathcal{W}))^m$ and so the inequalities hold.

(c) If a', a'' are the minimum and maximum integers in Σ, respectively then we set $C = \frac{1}{2}\log(a' + 1)$ and $D = \log(\frac{1}{2}(a'' + \sqrt{(a'')^2 + 4})) = \log \rho(\text{Mat}(a''))$. The result can be established by induction using the continuant properties. □

We now can state and prove what we will need to establish transcendence in the class of quasi-periodic continued fractions.

Lemma 11. *If* $\mathbf{u}$ *satisfies a condition* (w', w'') *with* $w', w'' > 0$ *with the accompanying triples* $(U_n, V_n, W_n)_{n\geqslant 1}$*, and if there exists a* $\delta > 0$ *such that* $\log K(W_n) > (1+\delta)\log K(U_n)$*, then* $\alpha = [0, \mathbf{u}]$ *is transcendental.*

Proof. We first prove that there exists $\eta > 0$ such that

$$\log K(W_n) > (1+\eta)\log K(U_n) + \eta \log K(U_n V_n)$$

for all sufficiently large n.

By Lemma 10(c) and condition (b, iv) of Definition 4

$$\log K(V_n) < D|V_n| \leqslant \frac{D}{w''}|W_n| < \frac{D}{Cw''}\log K(W_n).$$

Then

$$\begin{aligned}
&\log K(W_n) - (1+\eta)\log K(U_n) - \eta \log K(U_n V_n) \\
&\quad > \log K(W_n) - (1+\eta)\log K(U_n) - \eta\big(\log K(U_n) + \log K(V_n) + \log 2\big) \\
&\quad > \left(1 - \frac{\eta D}{Cw''}\right)\log K(W_n) - (1+2\eta)\log K(U_n) - \eta\log 2 \\
&\quad = \left(1 - \frac{\eta D}{Cw''}\right)\left[\log K(W_n) - \frac{1+2\eta}{(1-\frac{\eta D}{Cw''})}\log K(U_n)\right] - \eta \log 2 \\
&\quad > \left(1 - \frac{2D}{Cw''}\left[1+\delta - \frac{1+2\eta}{1-\frac{\eta D}{Cw''}}\right]\right)\log K(U_n) - \eta\log 2.
\end{aligned}$$

So if $\eta > 0$ is chosen so that $\eta < \frac{Cw''}{D}$ and $\frac{1+2\eta}{1-\frac{\eta D}{Cw''}} < 1 + \frac{\delta}{2}$, then

$$\log K(W_n) - (1+\eta)\log K(U_n) - \eta\log K(U_n V_n) > \frac{\delta}{2}\log K(U_n) - \eta \log 2 > 0$$

for $n \geqslant n_0$, say.

Also

$$\begin{aligned}
&(1+\eta)(\log q_{r_n} + \log q_{r_n+s_n}) - \log(q_{r_n+s_n+t_n}) \\
&\quad = (1+\eta)\log K(U_n) + (1+\eta)\log K(U_n V_n) - \log K(U_n V_n W_n) \\
&\quad < (1+\eta)\log K(U_n) + (1+\eta)\log K(U_n V_n) - \log K(U_n V_n) - \log K(W_n) \\
&\quad < 0 \quad \text{for } n \geqslant n_0.
\end{aligned}$$

So $\frac{(q_{r_n} q_{r_n+s_n})^{1+\eta}}{q_{r_n+s_n+t_n}} < 1$ as required for transcendence [1]. $\square$

4. Transcendence

If $\mathbf{u} \in \Sigma^\omega$ and $\alpha = [0, \mathbf{u}]$ is the associated continued fraction with convergents $(\frac{p_n}{q_n})_{n \geqslant 0}$, then we set $m = \liminf_{n\to\infty} q_n^{1/n}$ and $M = \limsup_{n\to\infty} q_n^{1/n}$. Note that $M < \infty$, since by definition the sequence $\mathbf{u}$ only takes finitely many values.

A quasi-periodic infinite word, $\mathbf{u}_\lambda$, having the property that

$$\limsup_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} = \infty$$

may not satisfy a condition (w', w''). In order to resolve this possibility we can make use of the existing Theorem 2 of Baker [5].

Lemma 12. *Suppose $\lambda_n > \beta\lambda_{n-1}$ for $n \geqslant 2$ where $\beta > 1$. If $\mathbf{u} = \mathbf{u}_\lambda$ $(w_1, w_2, \ldots, w_k)$ is a non-trivial quasi-periodic infinite word, set $L = \frac{\beta}{\beta-1}(2\frac{\log M}{\log m} - 1)$. Then if $\limsup_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} > L$, $\alpha = [0, \mathbf{u}]$ is transcendental.*

Proof. By assumption there is an integer i, $1 \leqslant i \leqslant k$, and a subsequence $(n_j)_{j\geqslant 1}$ of integers with $n_j \equiv i \bmod k$ for $j \geqslant 1$, such that $\lambda_{n_j} > L\lambda_{n_j-1}$.

Put $x_{n_j} = w_1^{\lambda_1} w_2^{\lambda_2} \ldots w_{i-1}^{\lambda_{n_j-1}}$, with the usual convention that if $i = 1$ we replace w_0 by w_k. Then $x_{n_j} w_i^{\lambda_{n_j}}$ is a prefix of $\mathbf{u}$. As in Lemma 6 we note that $\lambda_{n_j-t-1} < \frac{1}{\beta^t}\lambda_{n_j-1}$ for $1 \leqslant t \leqslant n_j - 1$. Thus $|x_{n_j}| < (1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{n_j-2}})\lambda_{n_j-1}$ for $1 \leqslant t \leqslant n_j - 1$. So

$$|x_{n_j}| < \left(1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{n_j-2}}\right)\lambda_{n_j-1} \max_{1\leqslant t\leqslant k} |w_t|.$$

Hence

$$\frac{|w_i^{\lambda_{n_j}}|}{|x_{n_j}|} > \frac{\lambda_{n_j}|w_i|}{(\frac{\beta}{\beta-1})\lambda_{n_j-1} \max_{1\leqslant t\leqslant k} |w_t|} > 2\frac{\log M}{\log m} - 1.$$

Baker's theorem now implies that $\alpha = [0, \mathbf{u}]$ is transcendental. $\square$

We can now prove the main result of this paper.

Theorem 13. *Let $\tau = \frac{1}{2}(1 + \sqrt{5})$ and $\delta > 0$. If $(\lambda_n)_{n\geqslant 1}$ is a sequence of positive integers satisfying $\lambda_n > (\tau + \delta)\lambda_{n-1}$ for all $n \geqslant 2$ and $\{w_1, w_2, \ldots, w_k\}$ is a finite set of non-empty words that do not all commute then*

$$\alpha = \left[0, \mathbf{u}_\lambda(w_1, w_2, \ldots, w_k)\right]$$

is transcendental.

Proof. By Proposition 3 and subsequent comments, $\mathbf{u} = \mathbf{u}_\lambda$ $(w_1, w_2, \ldots, w_k)$ is not ultimately periodic. As we are only concerned in the transcendence of α we can delete any fixed initial segment of the continued fraction. Thus, without loss of generality, we can assume the w_i's are labeled so that

$$\rho(\mathcal{W}_k) = \max_{1 \leqslant i \leqslant k} \rho(\mathcal{W}_i)$$

where $\mathcal{W}_i = \mathrm{Mat}(w_i)$ for $1 \leqslant i \leqslant k$.

With L as defined in Lemma 12 we have shown that α is transcendental when $\limsup_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} > L$. So we can therefore suppose that $\lambda_n \leqslant (L+1)\lambda_{n-1}$ for $n \geqslant 2$ (perhaps having to remove a further fixed initial segment). Then Lemma 6 can be applied with $\beta = \tau + \delta$ and $\gamma = L + 1$, to show that $\mathbf{u}$ satisfies a condition (w', w''), as defined in the lemma. By Lemma 11 we need only prove that there exists $\delta' > 0$ such that $\log K(W_n) < (1+\delta') \log K(U_n)$ for all sufficiently large n. Using continuant property (b) and Lemma 10 we get:

$$\begin{aligned}\log K(U_n) &< \log K\left(w_1^{\lambda_1}\right) + \log K\left(w_2^{\lambda_2}\right) + \cdots + \log K\left(w_{k-2}^{\lambda_{nk-2}}\right) + nk \log 2 \\ &< \left(\sum_{j=1}^{nk-2} \lambda_i\right) \log \rho(\mathcal{W}_k) + nk \log 2.\end{aligned}$$

Now

$$\sum_{j=1}^{nk-2} \lambda_i < \frac{1}{\beta(\beta-1)} \lambda_{nk},$$

so

$$\log K(U_n) < \frac{1}{\beta(\beta-1)} \lambda_{nk} \log \rho(\mathcal{W}_k) + nk \log 2.$$

On the other hand,

$$\log K(W_n) \geqslant \log K\left(w_k^{\lambda_{nk}}\right) > \lambda_{nk} \log \rho(\mathcal{W}_k) - \log 2.$$

Since $\tau^2 = \tau + 1$ we obtain $\beta(\beta - 1) > 1 + 3\delta$ and hence

$$\frac{\log K(W_n)}{\log K(U_n)} > \frac{1 + 3\delta - (\frac{\log 2}{\lambda_{n-k}} \log \rho(\mathcal{W}_k))}{1 + (\frac{nk \log 2}{\lambda_{nk} \log \rho(\mathcal{W}_k)})} > 1 + \delta$$

for n sufficiently large, since (λ_n) has exponential growth. $\square$

We can get a better result in a couple of special cases. As the proof of the following theorem proceeds in a similar vein to Theorem 13 we will just briefly mention the essential points.

Theorem 14. *Suppose $(\lambda_n)_{n \geqslant 1}$ is a sequence of positive integers, $\delta > 0$ and $\lambda_n > (\sqrt{2} + \delta)\lambda_{n-1}$ for all $n \geqslant 2$. If $\mathbf{u}_\lambda(w_1, w_2, \ldots, w_k)$ is a non-trivial quasi-periodic infinite word, then the continued fraction $[0, \mathbf{u}_\lambda(w_1, w_2, \ldots, w_k)]$ is transcendental if either*

(a) $k = 2$ *or*
(b) $\rho(\mathcal{W}_i) = \rho(\mathcal{W}_k)$ *for* $1 \leqslant i \leqslant k$.

Proof. Put $\beta = \sqrt{2} + \delta$.

(a) We obtain

$$\log K(U_n) < \frac{1}{\beta^2 - 1}\big[\lambda_{2n-1} \log \rho(\mathcal{W}_1) + \lambda_{2n} \log \rho(\mathcal{W}_2)\big] + 2n \log 2.$$

Also $\log K(W_n) > \lambda_{2n-1} \log \rho(\mathcal{W}_1) + \lambda_{2n} \log \rho(\mathcal{W}_2) - \log 2$. Since $\beta^2 - 1 > 1 + 2\delta$ we can conclude as before.

(b) We obtain

$$\log K(U_n) < \frac{1}{\beta - 1} \lambda_{nk-1} \log \rho(\mathcal{W}_k) + nk \log 2.$$

Also

$$\begin{aligned} \log K(W_n) &> (\lambda_{nk-1} + \lambda_{nk}) \log \rho(\mathcal{W}_k) - \log 2 \\ &> (\beta + 1)\lambda_{nk-1} \log \rho(\mathcal{W}_k) - \log 2. \end{aligned}$$

Again

$$\beta + 1 > \frac{1 + 2\delta}{\beta - 1}$$

and the result follows. □

We conclude this section by giving a couple of examples of infinite words which have sub-affine complexity, but for which the new technique is not effective.

Example 15. Let $\mathbf{u} = a^{\lambda_1} b^{\lambda_2} a^{\lambda_3} b^{\lambda_4} \ldots$, where

$$\lambda_{n+2} = 2\lambda_n \quad \text{for } n \geqslant 1 \text{ and } 1 < \lambda_1 < \lambda_2 < 2\lambda_1.$$

Then $\mathbf{u}$ is quasi-periodic and using Proposition 7 we obtain the fact that $p(n) \leqslant 4n$. Indeed, with the notation of Proposition 7 we see that $\lambda_{m_0+2} = 2\lambda_{m_0} > \lambda_{m_0} + \lambda_{m_0-1} \geqslant n$, and hence $t_0 \leqslant 1$.

However we can also show that $\log K(W_n) < \log K(U_n)$, so that the fundamental inequality of Lemma 11 is not satisfied.

Example 16. Let $\mathbf{u} = a^{\lambda_1} b a^{\lambda_2} b a^{\lambda_3} \ldots$, where $(\lambda_n)_{n \geqslant 1}$ is the same sequence as in Example 15. It should be noted that $m = M$ in this case, as the density of a's equals 1 (see [4]). The complexity of $\mathbf{u}$ is given by $p(n) = 3n - 8$ for $n \geqslant 5$ yet the transcendence of $[0, \mathbf{u}]$ cannot be obtained by the present methods.

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